Decision Procedures
An Algorithmic Point of View
Linear Arithmetic

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ETH/Technion

Version 1.0, 2007

Part V

Linear Arithmetic

Fourier-Motzkin Variable Elimination
Outline

Fourier-Motzkin Variable Elimination

- Goal: decide satisfiability of conjunction of linear constraints over reals

$$\bigwedge_{1 \le i \le m} \sum_{1 \le j \le n} a_{i,j} x_j \le b_i$$

- Earliest method for solving linear inequalities
- Discovered in 1826 by Fourier, re-discovered by Motzkin in 1936
- Basic idea of variable elimination:
  - Pick one variable and eliminate it
  - Continue until all variables but one are eliminated

Linear Arithmetic over the Reals

Input: A system of conjoined linear inequalities $A\overline{x} \le \overline{b}$

m constraints

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{22} & \cdots & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \le \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{pmatrix}$$

n variables

Removing unbounded variables

- Iteratively remove variables that are not bounded in both ways (and all the constraints that use them)
- The new problem has a solution iff the old problem has one!

~~$8x \ge 7y$~~
~~$x \ge 3$~~
$y \ge z$
$z \ge 10$
$20 \ge z$

$\longrightarrow$

~~$y \ge z$~~
$z \ge 10$
$20 \ge z$

$\longrightarrow$

$z \ge 10$
$20 \ge z$

Partitioning the Constraints

1. When eliminating x_n, partition the constraints according to the coefficient a_{in}:
   - $a_{i,n} > 0$: upper bound β_i
   - $a_{i,n} < 0$: lower bound β_i

$$\sum_{j=1}^{n} a_{i,j} \cdot x_j \leq b_i$$

$$\Rightarrow \quad a_{i,n} \cdot x_n \leq b_i - \sum_{j=1}^{n-1} a_{i,j} \cdot x_j$$

$$\Rightarrow \quad x_n \leq \frac{b_i}{a_{i,n}} - \sum_{j=1}^{n-1} \frac{a_{i,j}}{a_{i,n}} \cdot x_j \quad =: \beta_i$$

Example for Upper and Lower Bounds

		Category?
(1)	$x_1 - x_2 \leq 0$	Upper bound
(2)	$x_1 - x_3 \leq 0$	Upper bound
(3)	$-x_1 + x_2 + 2x_3 \leq 0$	Lower bound
(4)	$-x_3 \leq -1$	

Assume we eliminate x_1.

Adding the constraints

2. For each pair of a lower bound $a_{l,n} < 0$ and upper bound $a_{u,n} > 0$, we have

$$\beta_l \quad \leq \quad x_n \quad \leq \quad \beta_u$$

3. For each such pair, add the constraint

$$\beta_l \quad \leq \quad \beta_u$$

Fourier-Motzkin: Example

			Category?
~~(1)~~	~~$x_1 - x_2 \leq 0$~~		Upper bound
~~(2)~~	~~$x_1 - x_3 \leq 0$~~		Upper bound
~~(3)~~	~~$-x_1 + x_2 + 2x_3 \leq 0$~~		Lower bound
(4)	$-x_3 \leq -1$		Lower bound
			we eliminate x_1
(5)	$2x_3 \leq 0$	(from 1,3)	Upper bound
(6)	$x_2 + x_3 \leq 0$	(from 2,3)	Upper bound
			we eliminate x_3
(7)	$0 \leq -1$	(from 4,5)	

$\rightarrow$ Contradiction (the system is UNSAT)

Complexity

- Worst-case complexity:

$$m \rightarrow m^2 \rightarrow (m^2)^2 \rightarrow \ldots \rightarrow m^{2^n}$$

- Heavy! So why is it so popular in verification?

- The bottleneck: case-splitting