Decision Procedures

An Algorithmic Point of View

Linear Arithmetic

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ETH/Technion

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Part V

Linear Arithmetic

- Goal: Decide satisfiability of conjunction of linear constraints over integers

$$\sum_{0 \le i \le n} a_i x_i \ge 0$$

- Original application:
  program optimizations done by a compiler
- Extension of *Fourier-Motzkin* variable elimination:
  - Pick one variable and eliminate it
  - Continue until all variables but one are eliminated

Normalize coefficients: divide by the GCD

$$8x + 6y \leq 0 \quad \longrightarrow$$

Normalize coefficients: divide by the GCD

$$8x + 6y \leq 0 \quad \longrightarrow \quad 4x + 3y \leq 0$$

Normalize coefficients: divide by the GCD

$$\begin{array}{lcl} 8x + 6y \leq 0 & \longrightarrow & 4x + 3y \leq 0 \\ 4y \geq 1 & \longrightarrow & \end{array}$$

Normalize coefficients: divide by the GCD

$$\begin{array}{lcl} 8x + 6y \leq 0 & \longrightarrow & 4x + 3y \leq 0 \\ 4y \geq 1 & \longrightarrow & y \geq \lceil 1/4 \rceil \end{array}$$

Normalize coefficients: divide by the GCD

'Tightening'

$$
\begin{array}{lcl}
8x + 6y \leq 0 & \longrightarrow & 4x + 3y \leq 0 \\
4y \geq 1 & \longrightarrow & y \geq \lceil 1/4 \rceil
\end{array}
$$

Normalize coefficients: divide by the GCD

'Tightening'

$$
\begin{array}{lcl}
8x + 6y \le 0 & \longrightarrow & 4x + 3y \le 0 \\
4y \ge 1 & \longrightarrow & y \ge \lceil 1/4 \rceil \\
3x + 3y = 2 & \longrightarrow &
\end{array}
$$

Normalize coefficients: divide by the GCD

'Tightening'

$$
\begin{array}{lcl}
8x + 6y \leq 0 & \longrightarrow & 4x + 3y \leq 0 \\
4y \geq 1 & \longrightarrow & y \geq \lceil 1/4 \rceil \\
3x + 3y = 2 & \longrightarrow & x + y = 2/3
\end{array}
$$

Normalize coefficients: divide by the GCD

'Tightening'

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\begin{array}{lcll}
8x + 6y \leq 0 & \longrightarrow & 4x + 3y \leq 0 & \\
4y \geq 1 & \longrightarrow & y \geq \lceil 1/4 \rceil & \\
3x + 3y = 2 & \longrightarrow & x + y = 2/3 & \longrightarrow \quad \text{UNSAT}
\end{array}
$$

Eliminate equalities

- Let x_i denote a variable and a_i its coefficient, for $i = 1, 2, \ldots$
- Use *substitution* if there is a variable with coefficient $a_i = 1$

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- Otherwise, pick variable x_k from an equality, and make a_k positive

- Define $a \mathbin{\widehat{\text{mod}}} b := a - b\lfloor a/b + 1/2 \rfloor$
- Let $m = a_k + 1$
- Note that $a_k \mathbin{\widehat{\text{mod}}} m = -1$

Eliminate equalities

- Create new variable σ and add:

$$m\sigma = \sum_i (a_i \mathbin{\widehat{\bmod}} m) x_i$$

- Solve for x_k:

$$x_k = -m\sigma + \sum_{i \neq k} a_k (a_i \mathbin{\widehat{\bmod}} m) x_i$$

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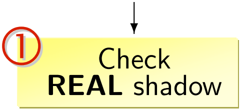
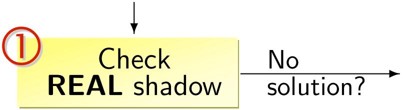
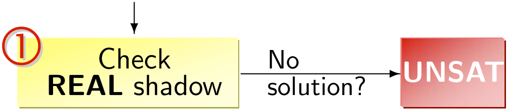
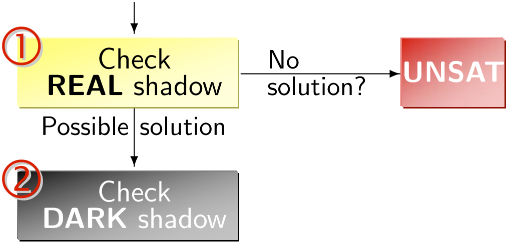
- Q: What is the point of adding a constraint to eliminate one?


1
Check
REAL shadow

Overview of the Omega Test




1
Check
REAL shadow
No
solution?
UNSAT


1
Check
REAL shadow
No
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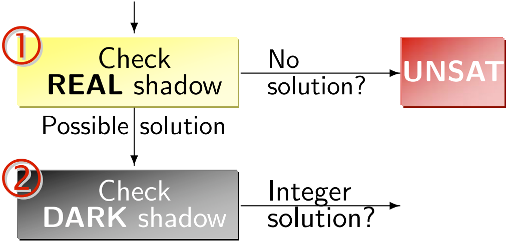
UNSAT
Possible solution
2
Check
DARK shadow


1
Check
REAL shadow
No
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UNSAT
Possible solution
2
Check
DARK shadow
Integer
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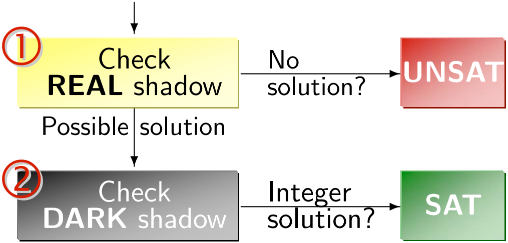
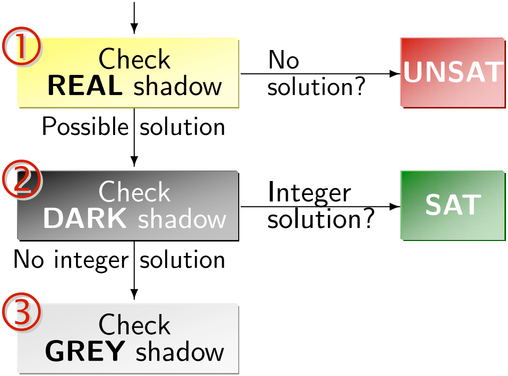
Overview of the Omega Test




1
Check
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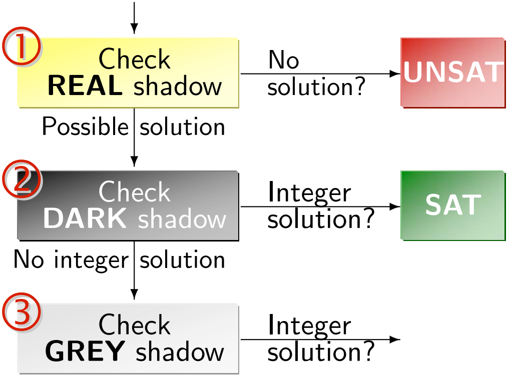
SAT
No integer solution
3
Check
GREY shadow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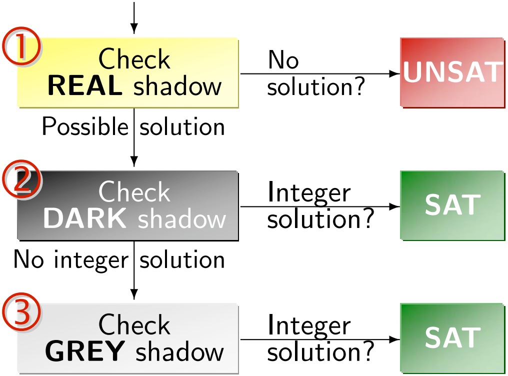
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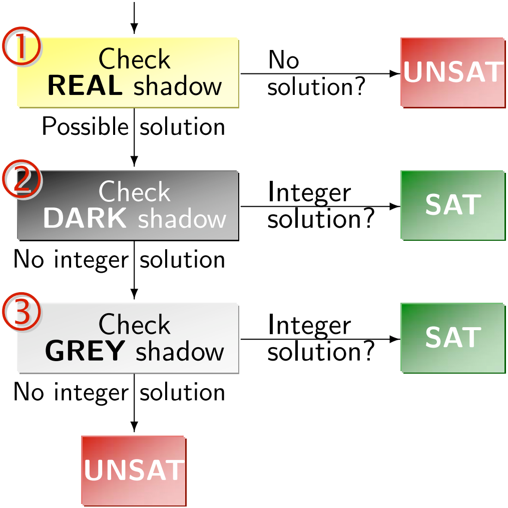
Overview of the Omega Test



1 Check
REAL shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\beta \quad \leq \quad bz \qquad az \quad \leq \quad \alpha$$

① Check **REAL** shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\begin{array}{rclcrcl} \beta & \leq & bz & \quad & az & \leq & \alpha \\ a\beta & \leq & abz & \quad & abz & \leq & b\alpha \end{array}$$

1 Check **REAL** shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

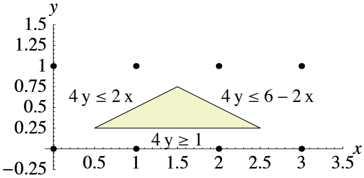
$$\begin{array}{rcl} \beta & \leq & bz \\ a\beta & \leq & abz \end{array} \qquad \begin{array}{rcl} az & \leq & \alpha \\ abz & \leq & b\alpha \end{array}$$

- Constraint for real shadow:

$$a\beta \leq b\alpha$$

- Add this constraint, and call Omega recursively!

1 Check **REAL** shadow

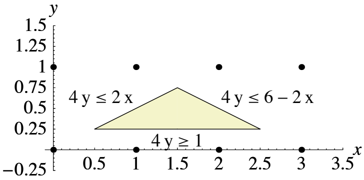


$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

1 Check **REAL** shadow



$4y \leq 2x$

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Eliminate x:

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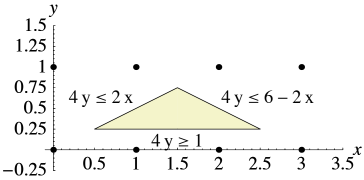

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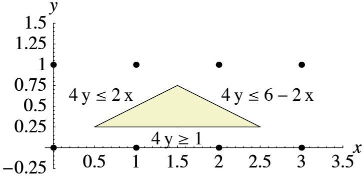
Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

$$4y \leq 6 - 4y$$

① Check **REAL** shadow



$4y \leq 2x$

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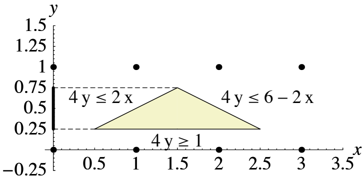
Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

$$4y \leq 6 - 4y$$
$$8y \leq 6$$

1 Check **REAL** shadow



$4y \leq 2x$

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Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

$$4y \leq 6 - 4y$$
$$8y \leq 6$$

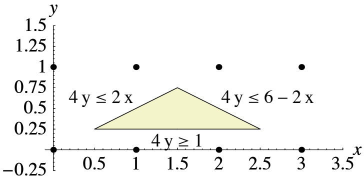
Real Shadow:

$8y \leq 6$, $4y \geq 1$ $\Rightarrow$ $y \leq 0.75$, $y \geq 0.25$

No integer solution

$\Longrightarrow$ Original problem has no solution

1 Check **REAL** shadow



$4y \leq 2x$

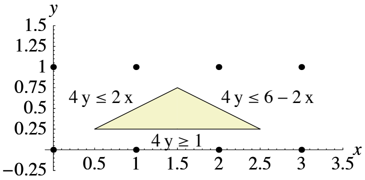
$4y \leq -2x + 6$

$4y \geq 1$

Let's eliminate y instead:

$1 \leq 4y$ $\quad$ $4y \leq 2x$

1 Check **REAL** shadow



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$$1 \leq 2x$$

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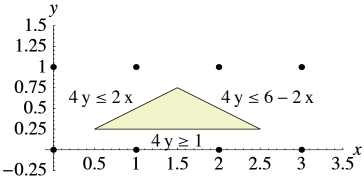

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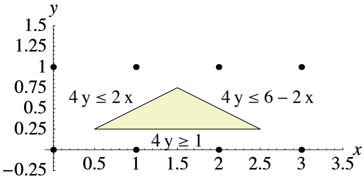

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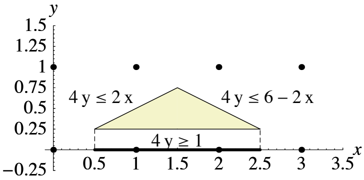
$1 \leq 4y$ $\quad$ $4y \leq -2x + 6$

$$1 \leq -2x + 6$$

The real shadow: Example II

① Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

Let's eliminate y instead:

$1 \leq 4y$ $\quad$ $4y \leq 2x$

$$1 \leq 2x$$

$1 \leq 4y$ $\quad$ $4y \leq -2x + 6$

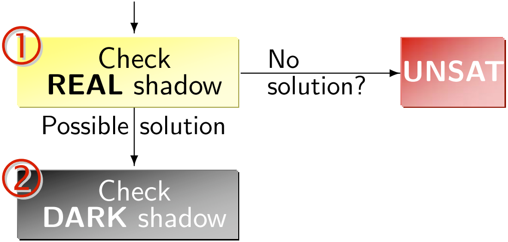
$$1 \leq -2x + 6$$

Real Shadow:

$$1 \leq 2x \qquad x \leq 0.5$$
$$1 \leq -2x + 6 \qquad x \geq 2.5$$

Integer solution!

But original problem has no integer solution!



- An integer solution for the REAL shadow does not guarantee that there is an integer solution for the original problem
- Thus, we check the DARK shadow next

② Check **DARK** shadow

- Idea of the DARK shadow:

$$\beta \quad \leq \quad bz \qquad\qquad\qquad az \quad \leq \quad \alpha$$

2 Check **DARK** shadow

- Idea of the DARK shadow:

$$\begin{array}{rclcrclcl} \beta & \leq & bz & \mid : b & az & \leq & \alpha & \mid : a & \\ \frac{\beta}{b} & \leq & z & & z & \leq & \frac{\alpha}{a} & & z \in \mathbb{N} \end{array}$$

② Check **DARK** shadow

- Idea of the DARK shadow:

$$\begin{array}{rclll} \beta & \leq & bz & \mid : b & \\ \frac{\beta}{b} & \leq & z & & \end{array} \qquad \begin{array}{rcll} az & \leq & \alpha & \mid : a \\ z & \leq & \frac{\alpha}{a} & \end{array} \qquad z \in \mathbb{N}$$

- How to compute the dark shadow?
- Try to *prove* that there is an integer z between $\frac{\beta}{b}$ and $\frac{\alpha}{a}$

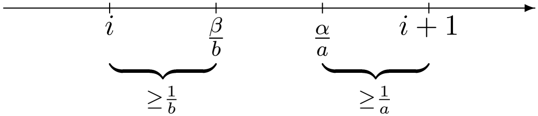
2 Check **DARK** shadow

Assume there is no integer z between $\frac{\beta}{b}$ and $\frac{\alpha}{a}$.

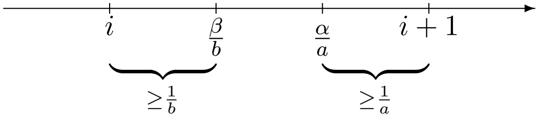
2 Check **DARK** shadow

Assume there is no integer z between $\frac{\beta}{b}$ and $\frac{\alpha}{a}$. Then:

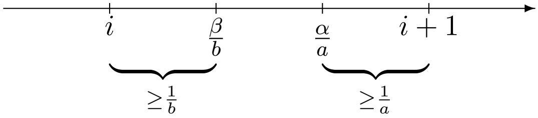
Let $i := \lfloor \frac{\beta}{b} \rfloor \qquad i \in \mathbb{Z}$



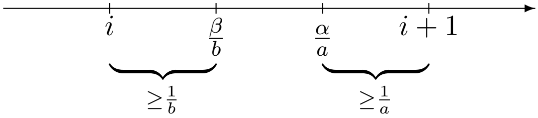
Dark shadow: Proof by contradiction



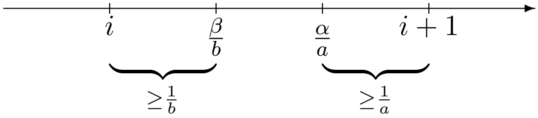
$$\begin{aligned} \frac{\beta}{b} - i &\geq \frac{1}{b} \\ i + 1 - \frac{\alpha}{a} &\geq \frac{1}{a} \end{aligned}$$



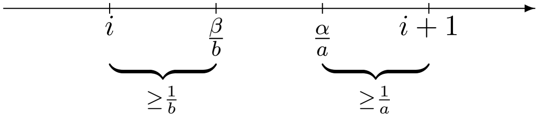
$$\begin{array}{rcl}
\frac{\beta}{b} - i & \geq & \frac{1}{b} \\
i + 1 - \frac{\alpha}{a} & \geq & \frac{1}{a} \\
\hline\hline
\frac{\beta}{b} + 1 - \frac{\alpha}{a} & \geq & \frac{1}{b} + \frac{1}{a}
\end{array}$$



$$\begin{array}{rcll}
\frac{\beta}{b} - i & \geq & \frac{1}{b} & \\
i + 1 - \frac{\alpha}{a} & \geq & \frac{1}{a} & \\
\hline\hline
\frac{\beta}{b} + 1 - \frac{\alpha}{a} & \geq & \frac{1}{b} + \frac{1}{a} & \quad | \cdot a \cdot b \\
a\beta + ab - b\alpha & \geq & a + b &
\end{array}$$



$$
\begin{array}{rcll}
\frac{\beta}{b} - i & \geq & \frac{1}{b} & \\
i + 1 - \frac{\alpha}{a} & \geq & \frac{1}{a} & \\
\hline\hline
\frac{\beta}{b} + 1 - \frac{\alpha}{a} & \geq & \frac{1}{b} + \frac{1}{a} & \quad | \cdot a \cdot b \\
a\beta + ab - b\alpha & \geq & a + b & \quad | - ab \\
a\beta - b\alpha & \geq & -ab + a + b &
\end{array}
$$



$$\begin{array}{rcll}
\frac{\beta}{b} - i & \geq & \frac{1}{b} & \\
i + 1 - \frac{\alpha}{a} & \geq & \frac{1}{a} & \\
\hline\hline
\frac{\beta}{b} + 1 - \frac{\alpha}{a} & \geq & \frac{1}{b} + \frac{1}{a} & \mid \cdot a \cdot b \\
a\beta + ab - b\alpha & \geq & a + b & \mid - ab \\
a\beta - b\alpha & \geq & -ab + a + b & \mid \cdot (-1) \\
 & & & \\
b\alpha - a\beta & \leq & ab - a - b &
\end{array}$$

- From previous slide:

$$b\alpha - a\beta \quad \leq \quad ab - a - b$$

- From previous slide:

$$\begin{array}{rrcl} & b\alpha - a\beta & \leq & ab - a - b \\ \iff & \neg(b\alpha - a\beta & > & ab - a - b) \end{array}$$

- From previous slide:

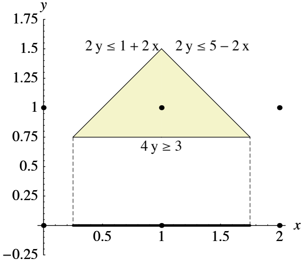
$$
\begin{array}{rrcl}
 & b\alpha - a\beta & \leq & ab - a - b \\
\iff & \neg(b\alpha - a\beta & > & ab - a - b) \\
\iff & \neg(b\alpha - a\beta & \geq & ab - a - b + 1)
\end{array}
$$

- From previous slide:

$$\begin{array}{rrcl} & b\alpha - a\beta & \leq & ab - a - b \\ \iff & \neg(b\alpha - a\beta & > & ab - a - b) \\ \iff & \neg(b\alpha - a\beta & \geq & ab - a - b + 1) \\ \iff & \neg(\underbrace{b\alpha - a\beta \;\; \geq \;\; (a-1)(b-1)}_{*}) \end{array}$$

- Thus, if * holds, we know that there must be an integer solution.
- If $a = 1$ or $b = 1$, then this is the same as the real shadow. This case is called an exact projection.

② Check **DARK** shadow

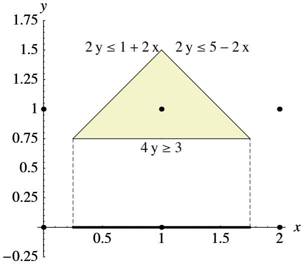


$$2y \leq 2x + 1$$

$$2y \leq -2x + 5$$

$$4y \geq 3$$

② Check **DARK** shadow



$$2y \leq 2x+1$$

$$2y \leq -2x+5$$

$$4y \geq 3$$

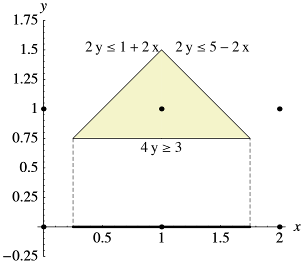
Eliminate y with the dark shadow:

$$2y \leq 2x+1 \qquad 4y \geq 3$$

$$4(2x+1) - 2 \cdot 3 \geq (2-1)(4-1)$$

Example for the dark shadow

2 Check **DARK** shadow



$2y \leq 2x + 1$

$2y \leq -2x + 5$

$4y \geq 3$

Eliminate y with the dark shadow:

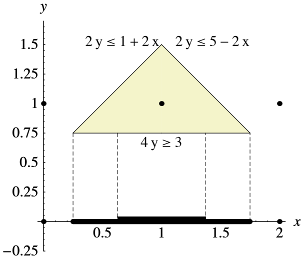
$2y \leq 2x + 1$ $\quad$ $4y \geq 3$

$$4(2x + 1) - 2 \cdot 3 \geq (2 - 1)(4 - 1)$$

$4y \geq 3$ $\quad$ $2y \leq -2x + 5$

$$4(-2x + 5) - 2 \cdot 3 \geq (2 - 1)(4 - 1)$$

② Check **DARK** shadow



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$$4(2x+1) - 2 \cdot 3 \geq (2-1)(4-1)$$

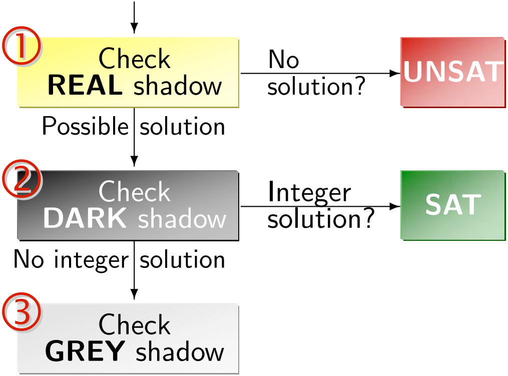
$4y \geq 3$ $\quad$ $2y \leq -2x + 5$

$$4(-2x+5) - 2 \cdot 3 \geq (2-1)(4-1)$$

Dark Shadow:

$$\begin{array}{rcl} x & \geq & 5/8 \\ x & \leq & 11/8 \end{array}$$

$\Longrightarrow$ Integer solution!



- No integer solution in the DARK shadow does not guarantee that there is no integer solution for the original problem
- Thus, we check the GREY shadow next

③ Check **GREY** shadow

Idea of the Grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

③ Check **GREY** shadow

Idea of the Grey shadow

If the real shadow R has integer solutions, but the dark shadow D does not, search $R \setminus D$.

$$\begin{array}{rl}
\text{In R:} & b\alpha \geq abz \geq a\beta \\
\textbf{Not}\text{ in D:} & ab - a - b \geq b\alpha - a\beta \\
\Longleftrightarrow & ab - a - b + a\beta \geq b\alpha \\
\Rightarrow & ab - a - b + a\beta \geq abz \geq a\beta
\end{array}$$

3 Check **GREY** shadow

Idea of the Grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

$$
\begin{array}{rl}
\text{In R:} & b\alpha \geq abz \geq a\beta \\
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\iff & ab - a - b + a\beta \geq b\alpha \\
\Rightarrow & ab - a - b + a\beta \geq abz \geq a\beta \qquad | : a \\
 & (ab - a - b)/a + \beta \geq bz \geq \beta
\end{array}
$$

3 Check **GREY** shadow

- Try all values of z such that

$$(ab - a - b)/a + \beta \geq bz \geq \beta$$

③ Check **GREY** shadow

- Try all values of z such that

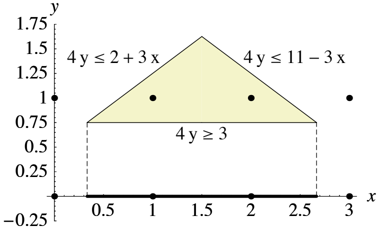
$$(ab - a - b)/a + \beta \geq bz \geq \beta$$

- Optimization: find the largest coefficient a in any upper bound and try the following for each lower bound $bz \geq \beta$:

$$bz = \beta + i \qquad \text{for } (ab - a - b)/a \geq i \geq 0$$

- As before, combine this with the original problem, and solve recursively.

3 Check **GREY** shadow

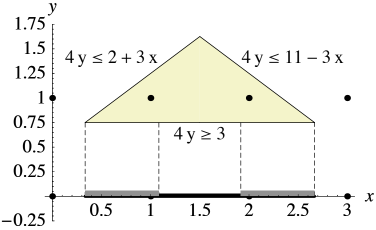


$$4y \leq 3x + 2$$

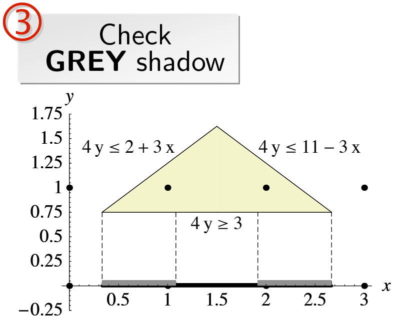
$$4y \leq -3x + 11$$

$$4y \geq 3$$

3 Check **GREY** shadow

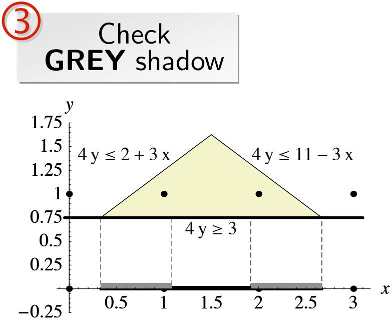


$$
\begin{aligned}
4y &\leq 3x + 2 \\
4y &\leq -3x + 11 \\
4y &\geq 3
\end{aligned}
$$



$$4y \leq 3x + 2$$
$$4y \leq -3x + 11$$
$$4y \geq 3$$

- Eliminate y:
  $a = 4,\ b = 4,\ \beta = 3$
- New constraint:
  $4y = 3 + i$ for
  $2 \geq i \geq 0$:



$$4y \leq 3x + 2$$

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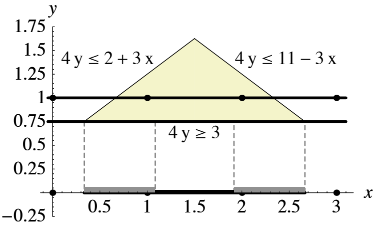
- Eliminate y:
  $a = 4,\ b = 4,\ \beta = 3$
- New constraint:
  $4y = 3 + i$ for
  $2 \geq i \geq 0$:

  $$4y = 3$$

3 Check **GREY** shadow



$$4y \leq 3x + 2$$
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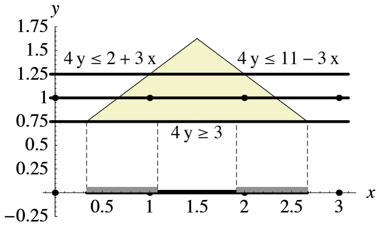
$$4y \geq 3$$

- Eliminate y:
  $a = 4,\ b = 4,\ \beta = 3$
- New constraint:
  $4y = 3 + i$ for
  $2 \geq i \geq 0$:
  $4y = 3$
  $4y = 4$

Example of the grey shadow

Check
GREY shadow



$$4y \leq 3x + 2$$

$$4y \leq -3x + 11$$

$$4y \geq 3$$

- Eliminate y:
  $a = 4,\ b = 4,\ \beta = 3$
- New constraint:
  $4y = 3 + i$ for
  $2 \geq i \geq 0$:

  $$4y = 3$$
  $$4y = 4$$
  $$4y = 5$$

$\Longrightarrow$ Integer solution with $4y = 4$