Decision Procedures
An Algorithmic Point of View
Linear Arithmetic

D. Kroening O. Strichman

ETH/Technion

Version 1.0, 2007

Part V

Linear Arithmetic

The Omega Test
Outline

The Omega Test

- Goal: Decide satisfiability of conjunction of linear constraints over integers

$$\sum_{0 \le i \le n} a_i x_i \ge 0$$

- Original application:
  program optimizations done by a compiler
- Extension of *Fourier-Motzkin* variable elimination:
  - Pick one variable and eliminate it
  - Continue until all variables but one are eliminated

Preprocessing (1)

Normalize coefficients: divide by the GCD

'Tightening'

$$\begin{array}{llll} 8x + 6y \le 0 & \longrightarrow & 4x + 3y \le 0 & \\ 4y \ge 1 & \longrightarrow & y \ge \lceil 1/4 \rceil & \\ 3x + 3y = 2 & \longrightarrow & x + y = 2/3 & \longrightarrow \text{UNSAT} \end{array}$$

Preprocessing (2)

Eliminate equalities

- Let x_i denote a variable and a_i its coefficient, for $i = 1, 2, \ldots$
- Use *substitution* if there is a variable with coefficient $a_i = 1$
- Otherwise, pick variable x_k from an equality, and make a_k positive

- Define $a \mathbin{\widehat{\bmod}} b := a - b\lfloor a/b + 1/2 \rfloor$
- Let $m = a_k + 1$
- Note that $a_k \mathbin{\widehat{\bmod}} m = -1$

Preprocessing (2)

Eliminate equalities

- Create new variable σ and add:

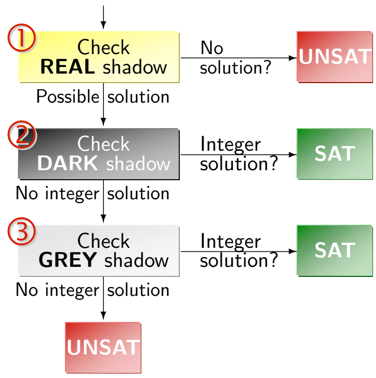
$$m\sigma = \sum_i (a_i \widehat{\bmod}\, m) x_i$$

- Solve for x_k:

$$x_k = -m\sigma + \sum_{i \neq k} a_k (a_i \widehat{\bmod}\, m) x_i$$

- Q: What is the point of adding a constraint to eliminate one?

Overview of the Omega Test



The real shadow

1 Check **REAL** shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\begin{array}{rclcrcl} \beta & \leq & bz & \quad & az & \leq & \alpha \\ a\beta & \leq & abz & & abz & \leq & b\alpha \end{array}$$

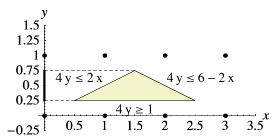
- Constraint for real shadow:

$$a\beta \leq b\alpha$$

- Add this constraint, and call Omega recursively!

The real shadow: Example I

1 Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

$4y \leq 6 - 4y$

$8y \leq 6$

Real Shadow:

$8y \leq 6$ $\quad y \leq 0.75$

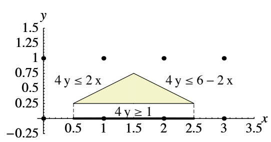
$4y \geq 1$ $\quad y \geq 0.25$

No integer solution

$\Longrightarrow$ Original problem has no solution

The real shadow: Example II

1 Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

Let's eliminate y instead:

$1 \leq 4y$ $\quad$ $4y \leq 2x$

$1 \leq 2x$

$1 \leq 4y$ $\quad$ $4y \leq -2x + 6$

$1 \leq -2x + 6$

Real Shadow:

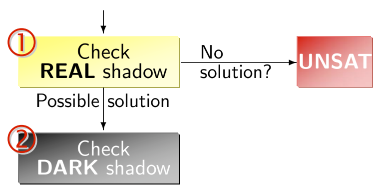
$1 \leq 2x$ $\quad x \leq 0.5$

$1 \leq -2x + 6$ $\quad x \geq 2.5$

Integer solution!

But original problem has no integer solution!

From Real to Dark Shadow



- An integer solution for the REAL shadow does not guarantee that there is an integer solution for the original problem
- Thus, we check the DARK shadow next

Idea of the dark shadow

② Check **DARK** shadow

- Idea of the DARK shadow:

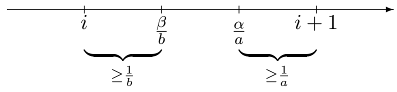
$$\begin{array}{rcll} \beta & \le & bz & \mid :b \\ \frac{\beta}{b} & \le & z & \end{array} \qquad \begin{array}{rcll} az & \le & \alpha & \mid :a \\ z & \le & \frac{\alpha}{a} & \end{array} \qquad z \in \mathbb{N}$$

- How to compute the dark shadow?
- Try to *prove* that there is an integer z between $\frac{\beta}{b}$ and $\frac{\alpha}{a}$

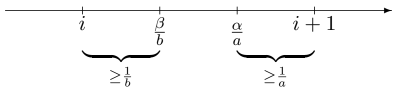
Dark shadow: Proof by contradiction

② Check **DARK** shadow

Assume there is no integer z between $\frac{\beta}{b}$ and $\frac{\alpha}{a}$. Then:
Let $i := \lfloor \frac{\beta}{b} \rfloor$ $\quad i \in \mathbb{Z}$



Dark shadow: Proof by contradiction



$$\begin{array}{rcll} \frac{\beta}{b} - i & \ge & \frac{1}{b} & \\ i + 1 - \frac{\alpha}{a} & \ge & \frac{1}{a} & \\ \hline \frac{\beta}{b} + 1 - \frac{\alpha}{a} & \ge & \frac{1}{b} + \frac{1}{a} & \mid \cdot a \cdot b \\ a\beta + ab - b\alpha & \ge & a + b & \mid -ab \\ a\beta - b\alpha & \ge & -ab + a + b & \mid \cdot (-1) \\ & & & \\ b\alpha - a\beta & \le & ab - a - b & \end{array}$$

Dark shadow: Proof by contradiction

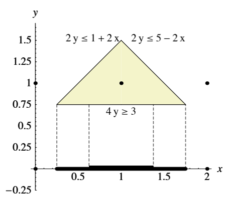
- From previous slide:

$$\begin{array}{rrcl} & b\alpha - a\beta & \le & ab - a - b \\ \iff & \neg(b\alpha - a\beta & > & ab - a - b) \\ \iff & \neg(b\alpha - a\beta & \ge & ab - a - b + 1) \\ \iff & \underbrace{\neg(b\alpha - a\beta \ge (a-1)(b-1))}_{*} \end{array}$$

- Thus, if * holds, we know that there must be an integer solution.
- If $a = 1$ or $b = 1$, then this is the same as the real shadow.
  This case is called an exact projection.

Example for the dark shadow

② Check **DARK** shadow



$2y \le 2x + 1$

$2y \le -2x + 5$

$4y \ge 3$

Eliminate y with the dark shadow:

$2y \le 2x + 1$ $\quad$ $4y \ge 3$

$4(2x+1) - 2 \cdot 3 \ge (2-1)(4-1)$

$4y \ge 3$ $\quad$ $2y \le -2x + 5$

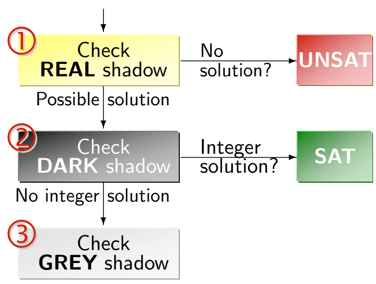
$4(-2x+5) - 2 \cdot 3 \ge (2-1)(4-1)$

Dark Shadow:

$$\begin{array}{rcl} x & \ge & 5/8 \\ x & \le & 11/8 \end{array}$$

$\Longrightarrow$ Integer solution!

From Dark to Grey Shadow



- No integer solution in the DARK shadow does not guarantee that there is no integer solution for the original problem
- Thus, we check the GREY shadow next

The grey shadow

③ Check **GREY** shadow

Idea of the Grey shadow

If the real shadow R has integer solutions, but the dark shadow D does not, search $R \setminus D$.

$$
\begin{array}{rl}
\text{In R:} & b\alpha \geq abz \geq a\beta \\
\textbf{Not}\text{ in D:} & ab - a - b \geq b\alpha - a\beta \\
\iff & ab - a - b + a\beta \geq b\alpha \\
\Rightarrow & ab - a - b + a\beta \geq abz \geq a\beta \quad | : a \\
 & (ab - a - b)/a + \beta \geq bz \geq \beta
\end{array}
$$

The grey shadow

③ Check **GREY** shadow

- Try all values of z such that

$$(ab - a - b)/a + \beta \geq bz \geq \beta$$

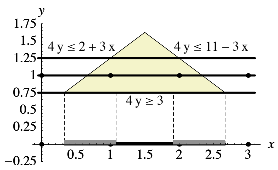
- Optimization: find the largest coefficient a in any upper bound and try the following for each lower bound $bz \geq \beta$:

$$bz = \beta + i \quad \text{for } (ab - a - b)/a \geq i \geq 0$$

- As before, combine this with the original problem, and solve recursively.

Example of the grey shadow

③ Check **GREY** shadow



$$
\begin{aligned}
4y &\leq 3x + 2 \\
4y &\leq -3x + 11 \\
4y &\geq 3
\end{aligned}
$$

- Eliminate y:
  $a = 4,\ b = 4,\ \beta = 3$
- New constraint:
  $4y = 3 + i$ for $2 \geq i \geq 0$:
  - $4y = 3$
  - $4y = 4$
  - $4y = 5$

$\Longrightarrow$ Integer solution with $4y = 4$