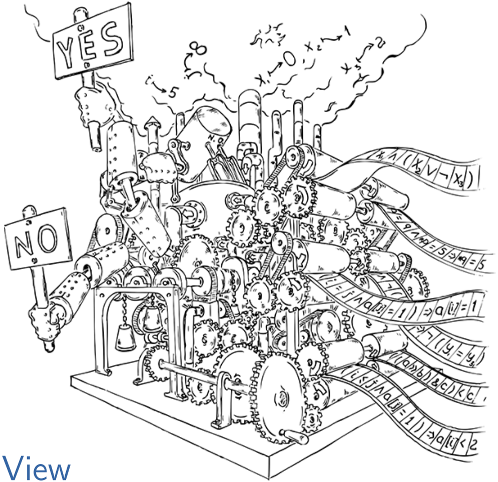
Gomory Cuts

Chapter 5

Linear Arithmetic



Decision Procedures

An Algorithmic Point of View

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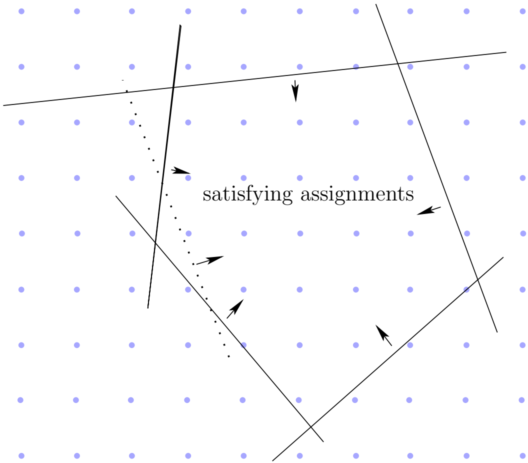
Revision 1.0

- Recall that in Branch & Bound we first solve a relaxed problem
  (i.e., no integrality constraints).
- We now study a method for adding *cutting planes* – constraints to the relaxed problem that do not remove integer solutions.
- Specifically, we will see Gomory cuts.

Cutting planes, geometrically.



The dotted line is a cutting plane.

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- ...and the solution α is

$$\{x_3 \mapsto 1.75, x_1 \mapsto 1, x_2 \mapsto 0.5\}$$

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- We now wish to rewrite this equation so the left-hand side is an integer:

$$x_3 - 1 = 0.75 + 0.5(x_1 - 1) + 2.5(x_2 - 0.5)\ . \tag{3}$$

- The two right-most terms must be positive because 1 and 0.5 are the lower bounds of x_1 and x_2, respectively.
- Since the right-hand side must add up to an integer as well, this implies that

$$0.75 + 0.5(x_1 - 1) + 2.5(x_2 - 0.5) \geq 1 \,. \tag{4}$$

Example: Gomory Cuts

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- This constraint is unsatisfied by α because $\alpha(x_1) = 1, \alpha(x_2) = 0.5$.
- Hence, this constraint removes the current solution.
- On the other hand, it is implied by the integer system of constraints, and hence cannot remove any *integer* solution.

- Generalizing this example:
  - Upper bounds.
  - Both positive and negative coefficients.

- The description that follows is based on
  - *Integrating Simplex with DPLL(T)*
    Technical report SRI-CSL-06-01
    Dutertre and de Moura (2006).

There are two preliminary conditions for deriving a Gomory cut from a constraint:

- The assignment to the basic variable has to be fractional.
- The assignments to all the nonbasic variables have to correspond to one of their bounds.

- Consider the i-th constraint:

$$x_i = \sum_{x_j \in \mathcal{N}} a_{ij} x_j \ , \tag{5}$$

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- Let α be the assignment returned by the general simplex algorithm. Thus,

$$\alpha(x_i) = \sum_{x_j \in \mathcal{N}} a_{ij} \alpha(x_j) \ . \tag{6}$$

- Partition the nonbasic variables to
  - those that are currently assigned their lower bound, and
  - those that are currently assigned their upper bound

$$\begin{aligned} J &= \{j \mid x_j \in \mathcal{N} \wedge \alpha(x_j) = l_j\} \\ K &= \{j \mid x_j \in \mathcal{N} \wedge \alpha(x_j) = u_j\}\,. \end{aligned} \tag{7}$$

- Subtracting (6) from (5) taking the partition into account yields

$$x_i - \alpha(x_i) = \sum_{j \in J} a_{ij}(x_j - l_j) - \sum_{j \in K} a_{ij}(u_j - x_j)\,. \tag{8}$$

- Let $f_0 = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$.
- As we assumed that $\alpha(x_i)$ is not an integer then $0 < f_0 < 1$.
- We can now rewrite (8) as

$$x_i - \lfloor \alpha(x_i) \rfloor = f_0 + \sum_{j \in J} a_{ij}(x_j - l_j) - \sum_{j \in K} a_{ij}(u_j - x_j) \; . \quad (9)$$

Note that the left-hand side is an integer.

We now consider two cases.

(Case 1)

- If

$$\sum_{j \in J} a_{ij}(x_j - l_j) - \sum_{j \in K} a_{ij}(u_j - x_j) > 0$$

then, since the right-hand side must be an integer,

$$f_0 + \sum_{j \in J} a_{ij}(x_j - l_j) - \sum_{j \in K} a_{ij}(u_j - x_j) \geq 1 \,. \tag{10}$$

(Still in case 1)

- We now split J and K as follows:

$$\begin{array}{rcl} J^+ & = & \{j \mid j \in J \wedge a_{ij} > 0\} \\ J^- & = & \{j \mid j \in J \wedge a_{ij} < 0\} \\ K^+ & = & \{j \mid j \in K \wedge a_{ij} > 0\} \\ K^- & = & \{j \mid j \in K \wedge a_{ij} < 0\} \end{array} \tag{11}$$

- Gathering only the positive elements in the left-hand side of (10) gives us:

$$\sum_{j \in J^+} a_{ij}(x_j - l_j) - \sum_{j \in K^-} a_{ij}(u_j - x_j) \geq 1 - f_0 \; , \tag{12}$$

or, equivalently,

$$\sum_{j \in J^+} \frac{a_{ij}}{1 - f_0}(x_j - l_j) - \sum_{j \in K^-} \frac{a_{ij}}{1 - f_0}(u_j - x_j) \geq 1 \; . \tag{13}$$

(Case 2)

- If

$$\sum_{j \in J} a_{ij}(x_j - l_j) - \sum_{j \in K} a_{ij}(u_j - x_j) \leq 0$$

then again, since the right-hand side must be an integer,

$$f_0 + \sum_{j \in J} a_{ij}(x_j - l_j) - \sum_{j \in K} a_{ij}(u_j - x_j) \leq 0\ . \tag{14}$$

Eq. (14) implies that

$$\sum_{j \in J^-} a_{ij}(x_j - l_j) - \sum_{j \in K^+} a_{ij}(u_j - x_j) \le -f_0 \ . \qquad (15)$$

Dividing by $-f_0$ gives us

$$-\sum_{j \in J^-} \frac{a_{ij}}{f_0}(x_j - l_j) + \sum_{j \in K^+} \frac{a_{ij}}{f_0}(u_j - x_j) \ge 1 \ . \qquad (16)$$

(End of case 2)

- Note that the left-hand side of both (13) and (16) is greater than zero.
- Therefore these two equations imply

$$\sum_{j \in J^+} \frac{a_{ij}}{1 - f_0}(x_j - l_j) - \sum_{j \in J^-} \frac{a_{ij}}{f_0}(x_j - l_j) + \sum_{j \in K^+} \frac{a_{ij}}{f_0}(u_j - x_j) - \sum_{j \in K^-} \frac{a_{ij}}{1 - f_0}(u_j - x_j) \geq 1 \; . \qquad (17)$$

- Since each of the elements on the left-hand side is equal to zero under the current assignment α, then α is ruled out by the new constraint.
- In other words: the solution to the linear problem augmented with the constraint is guaranteed to be different from the previous one.